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Outer curvature and conformal geometry of an imbedding

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The differential geometry of an imbedded (e.g. string or membrane world sheet) surface in a higher-dimensional background is shown to be conveniently describable (except in the null limit case) in terms of what are designated as its first, second, and third fundamental tensors, which will have the respective symmetry properties $\eta_{\mu\nu}=\eta_{(\mu\nu)}$ as a trivial algebraic identity, $K_{\mu\nu}{}^{\rho}=K_{(\mu\nu)}{}^{\rho}$ as the "generalised Weingarten identity", which is the (Frobenius type) integrability condition for the imbedding, and $\Xi_{\lambda\mu\nu}{}^{\rho}=\Xi_{(\lambda\mu\nu)}{}^{\rho}$ as a "generalised Codazzi equation", which depends on the background geometry being flat or of constant curvature, needing replacement by a more complicated expression for a generic value of the background curvature $B_{\kappa\lambda}{}^{\mu}{}_{\nu}$. The "generalised Gauss equation" expressing the dependence on this background curvature of the internal curvature tensor $R_{\kappa\lambda}{}^{\mu}{}_{\nu}$ of the imbedded surface is converted into terms of the first and second fundamental tensors, and it is thereby demonstrated that the vanishing of the (conformally invariant) "conformation tensor", i.e. the trace free part $C_{\mu\nu}{}^{\rho}$ of the second fundamental tensor $K_{\mu\nu}{}^{\rho}$, is a sufficient condition for conformal flatness of the imbedded surface (and thus in particular for the vanishing of its (Weyl type) conformal curvature tensor $C_{\kappa\lambda}{}^{\mu}{}_{\nu}$) provided the background is itself conformally flat. In a trio of which the first two members are the generalised Gauss and Codazzi equations, the "third" member is shown to give an expression in terms of $C_{\mu\nu}{}^{\rho}$ for the (trace free, conformally invariant) "outer curvature" tensor $\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}$, whose vanishing is the condition for feasibility of the natural generalisation of the Walker frame transportation ansatz. The vanishing of $C_{\mu\nu}{}^{\rho}$ is shown to be sufficient in a conformally flat background for the vanishing also of $\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}$.

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Dedicated to Roger Penrose

1. Introduction

The earliest work on differential geometry by Gauss and his contemporaries was concerned mainly with the extrinsic properties of an imbedding with respect to a background space. However, following the reorientation of the subject by Riemann, the emphasis has been increasingly redirected to the purely intrinsic properties of differential manifolds, to such an extent that textbooks dealing with differential geometry, particularly those most commonly consulted by theoretical physicists since the time of Einstein, have tended to neglect or even ignore the topic of imbeddings, which was the historic focus of interest. A revisit, and a more thorough exploration, of this older field of preoccupation has, however, been made overdue by many new developments, of which the most obviously notable is the rise of string theory and its generalisations to higher-dimensional systems. The purpose of this essay is to help fill the gap by making available to theoretical physicists a conveniently accessible self-contained account of some of the most important local curvature properties of differential imbeddings of arbitrary dimension p, say, in a background (Riemannian or pseudo-Riemannian) space of higher dimension n, say, including some (particularly the conformal) aspects that to my knowledge have not yet been competely treated even in the pure mathematical literature.

For students of earlier generations, including my own and that of Roger Penrose, to whom this work is dedicated, the most widely used introduction to this field was probably that provided by Eisenhart's classic textbook "Riemannian Geometry" [1], while for a rather younger generation an equivalent role was played, albeit using a very different notational style, by the "Foundations of Differential Geometry" of Kobyashi and Nomizu [2]. However, the source that was in practice by far the most helpful to me in obtaining the clarification offered here was Schouten's "Ricci Calculus" [3], while I should also mention the work of Chen [4,5] (in a more trendy notational style) as being, among the limited sample of relevant pure mathematical sources of which I am at present aware, one of those that goes furthest towards the results presented here.

The first part of this work, up to and including section 5 (and also the appendices) will be concerned with setting up a generally covariant background tensor notation scheme (and of an auxiliary frame system that will be used as sparingly as possible) and the systematic treatment of features arising at first differential order, generalising results whose historic prototypes are commonly cited with reference to the name of J. Weingarten. The second part (sections 6 to 10) is concerned with features arising at second differential order, including as a noteworthy example the relation that is the *third* in a set of which the first two members are much more widely known, being commonly cited with reference to the names of K.F. Gauss and D. Codazzi, respectively.

Already in Eisenhart's textbook [1] attention was explicitly drawn to the existence of a set of not two but *three* fundamental equations relating the geometrical configuration of an imbedding to the background spacetime curvature via the three relevant kinds (tangential, mixed, and orthogonal) of projection of the latter onto the imbedded surface. The first two of these projections give the pair of widely famed equations that can be considered as generalisations of results originally derived in the specialised context of hypersurfaces in a three-dimensional flat background by Gauss and Codazzi, but the last one has languished in relative obscurity. It is hoped that the present work will help to rectify this: it will be shown below that the "third equation" in question is expressible in a form (eq. 9.7) that reduces, for a flat, or at least conformally flat, background, to the by no means unmemorable formula

$$\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}=2C_{\sigma[\kappa}{}^{\mu}C_{\lambda]}{}^{\sigma}{}_{\nu}\,, \tag{1.1}$$

whose right hand side involves a conformally invariant and trace free "conformation tensor" $C_{\mu\nu}{}^{\rho}$ that will play a central role in all the work that follows – and whose vanishing will be shown to be a sufficient condition for conformal flatness of the imbedded surface – while its left hand side consists of the naturally defined and, as will be shown, conformally invariant and also entirely trace free "outer curvature tensor" $\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}$. The latter is not to be confused with the associated and again conformally invariant and trace free Weyl type "conformal curvature tensor" $C_{\kappa\lambda}{}^{\mu}{}_{\nu}$, which will also have to vanish under the same conditions, having also a quadratic dependence (eq. 10.7) on $C_{\mu\nu}{}^{\rho}$. The outer curvature tensor is interpretable as representing the Yang–Mills type gauge curvature of the bundle of surface orthogonal frames with respect to the connection that is naturally induced by the Riemannian structure. The vanishing of this tensor $\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}$ is thus the necessary and sufficient condition for it to be (locally) feasible to apply the natural generalisation of the Walker frame transportation ansatz [6] (on which Fermi–Walker type coordinate systems are based.)

One of the reasons why the "third" projection seems to have escaped the attention of early generations of geometers was that its result is always trivial – each side of (1.1) being identically zero – whenever either the dimension or the codimension of the imbedded surface is less than two, and hence always in a three-dimensional background, as also more generally for a curve (the case with which Walker was concerned) or for a hypersurface (the case with which Gauss and Codazzi were concerned) in a background of arbitrary dimension. When the background dimension is four the only non-trivial applications are therefore to two-dimensional surfaces (which includes the case of a string world sheet), and even this case is of relatively simple (Abelian) type, with the outer curvature determined (as described at the end of section 9) by just a single (pseudo-)scalar invariant Ω which, in the flat or conformally flat background case to which (1.1) applies, will be given in terms of the background measure tensor simply as

$$\Omega=\epsilon^{\kappa\lambda}{}_{\mu\nu}C_{\kappa\sigma}{}^{\mu}C_{\lambda}{}^{\sigma\nu}$$

(an equation which we shall leave unnumbered since, unlike the formally numbered equations throughout this work, its validity is dimensionally restricted); it will be made apparent in section 9 that (subject to suitable fixed boundary conditions) the surface integral of this quantity Ω will give a topological invariant that is the outer analogue of the well known Gauss–Bonnet invariant for the inner

(Ricci) curvature scalar (which means that it cannot give any effective contribution as a Lagrangian term in a variation principle). To obtain the most mathematically interesting (non-Abelian as opposed to Maxwellian type) examples, however, it is necessary to have a codimension of at least three, with a background dimension of five or more.

In accordance with the standard practice of calling things after famous early pioneers who may only have had a hazy glimmering of their existence or meaning, the illustrious name of G. Ricci was used by Schouten (as subsequently by Chen) for his version of the "third" projection from which our relation (1.1) was decanted. However, a generation earlier it had been given no name at all by Eisenhart (who, unlike Schouten and Chen, had apparently not yet clearly grasped the significance of the outer curvature). Eisenhart did, however, refer not only to Ricci's work on the subject at the beginning of the present century but also to that of a certain A. Voss, whose writing twenty years earlier might arguably qualify his name as a more appropriate label for the third equation. Whether one prefers to name the author who first considered it in a special case, or the author who first properly understood it in full generality, my impression (admittedly without having examined the original German and Italian language sources) is that Ricci (who in any case has so much else to his credit) should on this occasion miss out either way, and that, if it is not named after Voss, then the relation expressed here in the form (1.1) might more justifiably be named after Schouten himself.

What, as far as I know, is essentially new in the present work, in addition to what I hope will be perceived, at least by physicists, as a usefully streamlined but nevertheless sufficiently explicit representation scheme (based on imbedding supported background tensors, and intended to combine the advantages of the very different schemes used by Schouten and by Chen), is the consideration given to the conformal properties of the imbedding geometry, and in particular the attention that is drawn to the role of the conformally invariant trace free part $C_{\mu\nu}{}^{\rho}$ [defined by (5.9)] as distinct from the full second fundamental tensor $K_{\mu\nu}{}^{\rho}$ [defined by (5.2)], with which it is identifiable only in cases for which the imbedding is what in mathematical terminology would be referred to as an isometric harmonic [7,8] mapping, meaning that it satisfies a condition of stationarity with respect to small perturbations of the induced surface measure, which is equivalent to that of what has come to be known to theoretical physicists [9] as the Howe–Tucker [10] brane action.

2. Advantages of using imbedding supported background tensors

Although a plausibly sufficient reason in the case of earlier geometers, lack of interest in higher-dimensional cases can certainly not account for the failure to obtain relation (1.1) by a worker such as Eisenhart, who actually went to the trouble of writing down a dimensionally unrestricted and rather complicated ("Voss" or "Ricci" equation) relation that was logically equivalent to (1.1), but with additional terms on each side whose effect was to destroy their separate tensorial character, thus entirely obscuring the clear geometric interpretation (as a "Walker" or "Schouten" curvature equation) that becomes possible after the disparate contributions have been properly sorted out. Eisenhart's inability to see how to tidy up the mess was partly due to the unavailability at the time of concepts that have since been made widely familiar by the development of non-Abelian gauge theory, but it was also due to the use of an unnecessarily hairy notation system (of the kind that is still most commonly used by theoretical physicists) in which the use of too many kinds of (large and small, Latin and Greek, ...) indices made it hard to see the wood for the trees.

It is in reaction against the clumsy and inaesthetic notation schemes commonly used by their predecessors in the past (and by physicists far too often in the present) that pure mathematicians have more recently tended to go the opposite excess of abbreviating their terminology to such a degree that it becomes computationally powerless and potentially ambiguous outside its original context, refusing in extreme cases to admit the use of any indices at all on the ideological grounds that it is not merely inaesthetic but somehow actually immoral to introduce any machinery whose technicalities violate the underlying symmetries of the system under investigation. In this spirit one would not be allowed, for example, to express the second fundamental tensor by $K_{\mu\nu}{}^{\rho}$, as we do here, because the indices refer to an ultimately irrelevant choice of an underlying coordinate system. The trouble is that, if the offending tensor is just replaced by a "respectable" index-shorn operator symbol K, one is immediately faced with ambiguities of interpretation that oblige one to introduce explicit vectorial and covectorial arguments, which in this case would be a pair of (surface tangential) vectors X and Y, say, and a (surface orthogonal) covector λ, say, in terms of which K is specified by the numerical result of its action, which would be expressed as $K(X, Y)\lambda$, which is neither shorter nor more objective than the original expression $K_{\mu\nu}{}^{\rho}$, an irrelevant choice of local coordinates having been merely replaced by a no less irrelevant choice of basis (all that can be claimed is that either version is neater than the simultaneously base and coordinate dependent contraction formula for the corresponding scalar, which would of course have the form $K_{\mu\nu}{}^{\rho}X^{\mu}Y^{\nu}\lambda_{\rho}$).

It was a defensive counter reaction against the rather futile puritanism of the index suppression movement that Penrose developed the "abstract index" system [11,12], showing how one can get away with the use of what not only look like indices but actually *work* like indices, yet are nevertheless deemed actually not to be indices as far as their legal status is concerned. Of course this moral rehabilitation of indices does not in itself get round the original, purely pragmatic, problem of dealing with the complication that may arise when too many are involved: the management of such cases may require the use of sophisticated diagrammatic techniques [12], and the exercise of balanced judgement in deciding how far indices should be allowed to proliferate in particular circumstances before the advantages of explicitness are outweighed by the disadvantages of distracting technical complication.

The present work is intended not just to derive particular geometrical results, but also to demonstrate the practical advantages of analysing imbedded surfaces in terms of a formalism [13,14] whose use is implicit in (1.1) and whose guiding principle is to rely as much as possible on the use of *background* tensors, even for fields whose support is confined to the range of the imbedding. In this approach the use of coordinate and frame dependent quantities is designed to be a balanced compromise between, on one hand, the often inconvenient integrist inhibitions displayed, for example, by many followers of Kobyashi and Nomizu and, on the other hand, the unnecessary and distracting complication of notation that was exemplified to a moderate degree by Eisenhart and that has since been taken to far more extravagant extremes by many physicists (particularly those who prefer never to treat submanifolds as straightforward imbeddings but only as supports for Dirac distributions).

With the desideratum of a balanced compromise in view, the "imbedding supported background tensor" approach developed here is based on the use of tensorial indices that refer implicitly to (or, if they are interpreted as "abstract indices", that *simulate* reference to) a local coordinate patch on the *background* manifold. Frame indices are introduced when absolutely necessary, but they are systematically eliminated as soon as possible, the various contributions being systematically regrouped with the aim of getting to combinations that are tensorial in the strictest (generally covariant) sense. The distinguishing feature of the present approach is the scrupulous avoidance of the (occasionally useful but commonly abused) practice of introducing special coordinate systems (not to mention superfluous Dirac distributions) taylored to the imbedding of particular surfaces. The latter feature makes this formalism outstandingly effective for treating *intersections* in which several surfaces of dimension $p+1$, say, meet on a common boundary subsurface of dimension p. Such intersections are of frequent occurrence in the kind of physical (and in particular dynamical) applications for which the work of the following sections is primarily intended.

A noteworthy example, involving the kind of p-dimensional intersection to which the above considerations apply, is provided by the general purpose force balance equation governing the effect on each other's movement of mutual contact between several $(p+1)$-surface supported structures, each with its own (tangentially orientated) surface stress momentum–energy density tensor $\hat{T}^{\mu\nu}$, say. Its expression in most other schemes would be rather awkward, but in the "imbedding supported background tensor" notation used here it is given [14], in terms of the analogous (tangentially orientated) surface stress momentum–energy density tensor $T^{\mu\nu}$, say, of the p-dimensional junction structure, simply by

$$T^{\mu\nu}K_{\mu\nu}{}^{\rho}=\gamma^{\rho}{}_{\mu}\sum \hat{T}^{\mu\nu}\hat{\lambda}_{\nu}\,, \tag{2.1}$$

where the notation $\gamma^{\rho}{}_{\mu}$ and $K_{\mu\nu}{}^{\rho}$ is used for the (purely geometric) orthogonal projection tensor and second fundamental tensor as defined in the next sections [by (3.4) and (4.15), respectively], and where, in each term of the summation on the right, $\hat{\lambda}_{\nu}$ is the uniquely defined (tangentially orientated) unit normal from the junction p-surface into the corresponding member of the set of externally attached $(p+1)$-surfaces over which the summation is to be taken.

A typical application of the ubiquitously valid (but not correspondingly well-known) formula (2.1) is to the case (with $p=2$) of one of the string-like junctions between membrane world sheets in a connected cluster of soap bubbles. An equation of the same form (2.1) would also be applicable to a *junction between junctions*, which, in the case of a cluster of soap bubbles, would have the form of a timelike world line (with $p=1$) at the intersection of several (normally *four*) string-like membrane junctions. On the other hand, going back the other way towards higher dimension, (2.1) can also be applied to the case of an individual soap bubble membrane, considered as a "sail" (with $p=2$) whose motion is governed by the difference between the atmospheric gas contributions from either side. I have called (2.1) the "generalised sail equation" because its simplest applications (of which the most obvious is to the case of an ordinary nautical sail) are of this last type, in which the junction is just a *hypersurface*, which as such will have a unit normal λ_{ν} of its own that is unique up to a choice of sign, and for which the number of terms in the summation on the right will be *only two*, which means that the net contribution $\sum \hat{T}^{\mu\nu}\hat{\lambda}_{\nu}$ will be reducible to the form $[\hat{T}^{\mu\nu}]\lambda_{\nu}$, where $[T^{\mu\nu}]$ denotes just the ordinary *discontinuity* [7] between the "wind" stress momentum–energy density tensors on each side of the sail.

3. Local frames and the first fundamental tensor

Our ultimate intention here is to show how to set up a general purpose description of the most important local geometric structures, and in particular the various kinds of curvature that are associated with the imbedding of a spacelike or timelike p-surface in an n-dimensional space or spacetime background with metric $g_{\mu\nu}$, working as far as possible with generally covariant strictly tensorial quantities, with component indices μ, $\nu=1, \ldots, n$ associated with an arbitrary system of local coordinates x^{μ} on the n-dimensional background spacetime. As an auxiliary that is useful for the intermediate stages in many calculations, we shall, however, make use also of quantities that are only pseudo-tensorial in the sense of being dependent on the choice of a local orthonormal frame. We shall use Greek indices Λ, Θ, $\ldots=1, \ldots, n$ to label the chosen set of orthonormal frame basis vectors $\theta_A{}^\nu$ and their complementary forms $\theta^A{}_\nu$. Their orthonormality will be expressible by the condition that their contractions have the form

$$g_{\mu\nu}=\theta_{A\mu}\theta^A{}_\nu, \qquad g_{A\Theta}=\theta_{A\nu}\theta_\Theta{}^\nu\,, \tag{3.1}$$

on the understanding that the background (Lorentzian or Euclidean signature) metric $g_{\mu\nu}$ is used for raising and lowering spacetime indices, while the $g_{A\Theta}$ are the *constant* components of the corresponding frame component metric in standard (Minkowski or Cartesian) diagonal form.

As an essential aspect of our tactic (which distinguishes the present approach from much of the previously available literature on imbeddings) of working as far as possible only with quantities that are tensorial in the strictest sense, we shall rigorously avoid the introduction of any specialised coordinate system that is specifically adapted to the imbedding under consideration; in particular we shall eschew the use of any distinguished subset of p internal coordinates within the imbedded surface, such as are commonly introduced at the outset in traditional discussions of the subject. Although we shall thus work only with a single set of coordinates whose choice is completely arbitrary and unrelated to the imbedded surface, we shall, however, take account of the location of the imbedding in the choice of the auxiliary frame whenever it is required, taking it to be oriented so as to decompose the frame vectors as distinct subsets that are respectively *tangential* and *orthogonal* to the imbedded p-surface. Explicitly, the frame at each point will be taken to consist of two (mutually orthogonal) subsets, $\{\theta_A{}^\nu\}=\{\iota_A{}^\nu, \lambda_R{}^\nu\}$, of which the former "inner" subset, labelled by early Latin basis indices, A, B, $...=1, ..., p$, are tangential to the p-surface, while the latter "outer" subset, labelled by late Latin indices R, S, $...=p+1$, ..., n, are orthogonal to the p-surface, the mutual orthogonality of the two subsets of frame vectors being expressed by the condition $\iota_A{}^\nu\lambda_{R\nu}=0$.

The use of a preferentially oriented frame such as has just been described means that the original n-dimensional (Lorentzian or Euclidean) rotation group arbitrariness in the choice of frame will be broken, so as to leave a more restricted group consisting of the direct product of a p-dimensional "inner" rotation group acting within the tangent plane of the imbedding and of an $(n-p)$-dimensional "outer" rotation group acting orthogonally. One of the main tasks of this work will be to investigate the two distinct kinds of (respectively "inner" and "outer") curvature associated with these two ("inner" and "outer") frame gauge groups, the curvature itself being representable, as will be shown, in strictly tensorial (frame gauge independent) form.

Having set up any such surface adapted frame, we can immediately use it for the explicit construction of the obviously *frame independent*, and thus strictly tensorial quantity with general coordinate components $\eta^{\mu\nu}$ that we refer to [13,14] as the (first) *fundamental tensor* of the p-surface, which is obtained by summing over the inner frame vectors in the form

$$\eta^{\mu\nu} = \iota_A{}^{\mu}\iota^{A\nu} . \tag{3.2}$$

This field (which fully determines both the induced metric on the p-surface and the directional orientation of the tangent plane of its imbedding) is the natural starting point for any strictly tensorial analysis of the geometry of the imbedding. It is evident that the pth-rank operator of metric projection onto the p-dimensional tangent subspace to the imbedding is represented by the (conformally invariant) mixed form $\eta_\mu{}^\nu$ of the fundamental tensor, which is characterised algebraically by the symmetry and projection operator properties

$$\eta_{[\mu\nu]} = 0 , \qquad \eta^\mu{}_\nu \eta^\nu{}_\rho = \eta^\mu{}_\rho , \qquad \eta^\nu{}_\nu = p , \tag{3.3}$$

in which, as throughout, we employ the usual convention of using square and round brackets for index antisymmetrisation and symmetrisation, respectively.

The same information (in conjunction with the background metric) as is contained in the fundamental tensor is also contained in its *orthogonal complement*,

$$\gamma_{\mu\nu} = \lambda_{R\mu}\lambda^R{}_\nu , \tag{3.4}$$

which has the analogous properties

$$\gamma_{[\mu\nu]} = 0 , \qquad \gamma^\mu{}_\nu \gamma^\nu{}_\rho = \gamma^\mu{}_\rho , \qquad \gamma^\nu{}_\nu = n - p , \tag{3.5}$$

and whose (conformally invariant) mixed form $\gamma^\mu{}_\nu$ is evidently the $(n-p)$th-rank operator of tensorial projection orthogonal to the tangent plane of the p-surface. These two tensors are evidently related by the complementarity and mutual orthogonality relations

$$\eta^\mu{}_\nu + \gamma^\mu{}_\nu = g^\mu{}_\nu , \qquad \eta^\mu{}_\nu \gamma^\mu{}_\rho = 0 , \tag{3.6}$$

where $g^\mu{}_\nu$ is of course the (not just conformally but absolutely invariant) n-dimensional unit matrix.

4. Connection coefficients and the extrinsic imbedding curvature

As far as the n-dimensional background geometry is concerned, it is well known that there is no non-trivial purely tensorial basis independent quantity that can be constructed from the metric at first differential order; the frame connection, as defined most simply by

$$\beta_\mu{}^\nu{}_A = \nabla_\mu \theta_A{}^\nu \tag{4.1}$$

(where ∇_μ is the usual Riemannian covariant differentiation operator associated with $g_{\mu\nu}$) can of course be converted into *pseudo-tensorial* form by contraction with the frame vectors so as to obtain

$$\beta_\lambda{}^\mu{}_\nu = \theta^A{}_\nu \nabla_\lambda \theta_A{}^\mu , \tag{4.2}$$

but despite the elimination of the frame indices the resulting antisymmetric connection "tensor"

$$\beta_{\lambda\mu\nu} = \beta_{\lambda(\mu\nu)} \tag{4.3}$$

nevertheless remains *frame gauge dependent*, albeit to a somewhat lesser extent than the corresponding pure frame components, namely the rotation coefficients, which are expressible as

$$\beta_{A}{}^{\Theta}{}_{\Phi} = \theta^{\Theta}{}_{\nu} \nabla_{A} \theta_{\Phi}{}^{\nu} , \tag{4.4}$$

where the transformation between coordinate and frame components is performed by the usual contraction mechanism as exemplified in this case by the definition

$$\nabla_{A} = \theta_{A}{}^{\nu} \nabla_{\nu} . \tag{4.5}$$

The natural decomposition allowed by the preferential orientation of the frames with respect to an imbedding as described in the previous section allows the specification of operations of purely *tangential* covariant differentiation within the imbedded surface, in terms of the corresponding purely *internal* covariant differentiation operations

$$\nabla_{A} = \iota_{A}{}^{\nu} \nabla_{\nu} . \tag{4.6}$$

We thus can go on the define the corresponding purely *internal rotation coefficients* [the anologues within the surface of the background rotation coefficients (3.4)] according to the analogous specification

$$\rho_{A}{}^{B}{}_{C} = \iota^{B}{}_{\nu} \nabla_{A} \iota_{C}{}^{\nu} , \tag{4.7}$$

these quantities being interpretable as the internal frame components of the natural (Cartan–Riemann type) gauge connection for the group of p-dimensional internal frame rotations that preserve the constant unit diagonal form of the internal frame metric

$$\eta_{AB} = \iota_{A}{}^{\nu} \iota_{B\nu} , \tag{4.8}$$

which is to be used for raising and lowering of the internal frame indices A, B, C, ... The point to be emphasized (since it is what enables us to analyse the geometry of the p-surface without the need to complicate our notation scheme by the use of an auxiliary subsystem of specially chosen internal coordinates) is that the quantities η_{AB} and $\rho_{A}{}^{B}{}_{C}$ constructed above are the *same* as would have been obtained by the more traditional approach of first working out the p-dimensional metric induced directly in the surface by imbedding and then defining the operations of covariant differentiation on the tangential frame vectors in terms just of the ordinary p-dimensional Riemannian connection associated with this metric within the surface. This is because the operation (4.6) not only has the property of being well defined when acting on an arbitrarily oriented field that is differentiably supported by the p-surface (even if the field is not defined elsewhere in the neighbourhood), but moreover, when it acts on a vector that is restricted to be *tangential* to the surface, so that an alternative definition in terms only of the internal geometry is available, then it is easily verifiable that the effect of (4.6) as defined above will be the *same* as if the alternative purely intrinsic definition had been used.

It is evident that the tangential differentiation operators specified by (4.6) can also be used in a rather similar way to construct what may appropriately be defined as *external rotation coefficients*, according to the specification

$$\omega_A{}^R{}_S = \lambda^R{}_\nu \nabla_A \lambda_S{}^\nu , \tag{4.9}$$

these quantities being interpretable as the frame components of the naturally induced (Yang–Mills type) gauge connection for the group of $(n-p)$-dimensional external frame rotations that preserve the constant unit diagonal form of the external frame metric

$$\gamma_{RS} = \lambda_R{}^\nu \lambda_S{}^\nu . \tag{4.10}$$

In addition to the purely inner and purely outer frame rotation coefficients specified by (4.7) and (4.9), the p-surface adapted tangential and orthogonal basis vectors $\iota_A{}^\nu$ and $\lambda_R{}^\nu$ can be used for the definition of the *mixed* rotation coefficients, which are given by

$$K_{AB}{}^R = \lambda^R{}_\nu \nabla_A \iota_B{}^\nu = -\iota_B{}^\nu \nabla_A \lambda^R{}_\nu . \tag{4.11}$$

Although formally tensorial, the spacetime component versions (as obtained by contraction with the corresponding frame vectors) of the purely inner and purely outer rotation coefficients are still frame dependent; they are obtainable from the analogous spacetime component version $\beta_\lambda{}^\mu{}_\nu$ of the background connection by taking the contractions

$$\rho_\lambda{}^\mu{}_\nu = \eta_\lambda^\rho \eta_\sigma{}^\mu \eta_\nu{}^\tau \beta_\rho{}^\sigma{}_\tau , \qquad \omega_\lambda{}^\mu{}_\nu = \eta_\lambda^\rho \gamma_\sigma{}^\mu \gamma_\nu{}^\tau \beta_\rho{}^\sigma{}_\tau , \tag{4.12}$$

which evidently satisfy

$$\rho_{\lambda(\mu\nu)} = 0 , \qquad \gamma_\lambda{}^\sigma \rho_{\sigma\mu\nu} = \gamma_\mu{}^\sigma \rho_{\lambda\sigma\nu} = 0 , \tag{4.13}$$

$$\omega_{\lambda(\mu\nu)} = 0 , \qquad \gamma_\lambda{}^\sigma \omega_{\sigma\mu\nu} = \eta_\mu{}^\sigma \omega_{\lambda\sigma\nu} = 0 . \tag{4.14}$$

The frame gauge dependence of the internal and external frame rotation pseudotensors $\rho_{\lambda(\mu\nu)}$ and $\omega_{\lambda(\mu\nu)}$ means that they can always be set equal to zero at any single chosen point by an appropriate choice of the relevant frames. Nevertheless it will not in general be possible for them to be taken to vanish over the whole neighbourhoods of surfaces of arbitrary dimension, which would correspond to the implementation of higher-dimensional analogues of the Walker frame propagation ansatz (on which the construction of Fermi–Walker coordinate systems is based) [6], which is feasible automatically in the case of a one-dimensional trajectory, but which would in general break down except under the restrictive conditions required for vanishing of the relevant curvature tensors as evaluated in the following sections.

In contrast with the situation that has just been described, there is no problem of frame dependence for the spacetime component version of the mixed rotation coefficients (4.11), which is obtainable directly from the background connection coefficients by taking the contraction

$$K_{\lambda\mu}{}^{\nu}=\eta_\lambda{}^{\rho}\eta_\mu{}^{\tau}\gamma_\sigma{}^{\nu}\beta_\rho{}^{\sigma}{}_{\tau}\,. \tag{4.15}$$

This *extrinsic imbedding curvature tensor* evidently has the properties of tangentiality of the first two indices and orthogonality of the last,

$$\gamma_\lambda{}^{\sigma}K_{\sigma\mu}{}^{\nu}=\gamma_\mu{}^{\sigma}K_{\lambda\sigma}{}^{\nu}=K_{\mu\nu}{}^{\sigma}\eta_\sigma{}^{\nu}=0\,. \tag{4.16}$$

The fact that (4.15) determines a quantity that is strictly tensorial [and thus on a different footing from those defined by (4.12), which are merely pseudo-tensorial] is advertised by its award of a capital Latin (rather than small Greek) symbol, and will be made manifest by the alternative definition [13,14] to be given in the next section.

For poetical and historical reasons, and also as a matter of practical convenience, it is customary to introduce nomenclature based on the names of the pioneering precursors in any field. In such a scheme $\rho_{\lambda(\mu\nu)}$ might appropriately be referred to as the *Riemann pseudo-tensor* while $\omega_{\lambda(\mu\nu)}=0$ might correspondingly be referred to as the *Walker pseudo-tensor*. Going on in the same spirit (of naming something after someone who had only a first glimmering of its existence) one might appropriately refer to $K_{\mu\nu}{}^{\rho}$ as the *Weingarten tensor*, in honour of the famous early investigator of the differential properties of normals to a hypersurface [15–17].

5. The second fundamental tensor and its Weingarten identity

The strict *frame independence* of the imbedding curvature tensor $K_{\mu\nu}{}^{\rho}$ can be seen from the alternative, *manifestly* gauge independent construction in terms of the *tangential covariant differentiation* operator

$$\bar{\nabla}_\mu=\eta_\mu{}^{\rho}\nabla_\rho\,. \tag{5.1}$$

In terms of this operator a tensor that can easily be verified to be the same quantity as that given by (4.15) is specifiable directly in terms of the first fundamental tensor by

$$K_{\lambda\mu}{}^{\nu}=\eta_\mu{}^{\sigma}\bar{\nabla}_\lambda\eta_\sigma{}^{\nu}\,. \tag{5.2}$$

This tensor is thus appropriately describable as the *second fundamental tensor* since it contains complete algebraic information about the tangential derivative of the first fundamental tensor $\eta_{\mu\nu}$, the part that is projected out in (5.2) being recoverable by a symmetrisation operation:

$$\bar{\nabla}_\lambda \eta_{\mu\nu} = 2K_{\lambda(\mu\nu)} \,. \tag{5.3}$$

As well as being characterised generally by the obvious algebraic properties (4.16) of tangentiality of the first two indices and orthogonality of the last, this second fundamental tensor has the (generically) non-trivial symmetry property

$$K_{[\mu\nu]}{}^{\rho} = 0 \,, \tag{5.4}$$

which can be seen to follow from the consideration that, for any pair of given frame index values A and B, say, the commutator field $\nabla_A \iota_B{}^\mu - \nabla_B \iota_A{}^\mu$ of the corresponding tangential vector fields $\iota_A{}^\mu$ and $\iota_B{}^\mu$ must itself necessarily be tangential to the imbedded p-surface. Relation (5.4) is thus interpretable [13,14] (except in the one-dimensional case of a curve, for which it holds as a trivial identity) as a necessary and sufficient Frobenius type [3] integrability condition for the range of subspaces of the fundamental projector $\eta_\mu{}^\nu$ to mesh together to form a well-defined imbedded p-surface. In the special case of a hypersurface ($p=n-1$), the symmetry property (5.4) reduces to the equivalent of what is referred to by Hicks [16] (in the index expurgated language of mathematical integrism) as the "self-adjointness of the Weingarten mapping". It therfore seems appropriate to refer to the unrestricted version (5.4) as the generalised Weingarten identity, or, to do greater historical justice, as the *Weingarten–Frobenius identity*; it can be thought of as being on a similar footing with respect to the three-index imbedding curvature tensor as the more widely familiar Ricci symmetry property [see (7.8) below] of an ordinary four-index Riemannian curvature.

It is apparent from (3.11) that the first fundamental tensor can be considered as determining the orthogonally projected part of the acceleration vector $\dot{u}^\rho$ of the unit normalised tangent vector u^μ to any non-null curve in the p-surface according to the formula

$$\gamma^\rho{}_\nu \dot{u}^\nu = u^\mu u^\nu K_{\mu\nu}{}^\rho \,, \tag{5.5}$$

using the standard notation

$$\dot{u}^\rho = u^\mu \nabla_\mu u^\rho \,, \qquad u^\nu u_\nu = \mp 1 \,, \tag{5.6}$$

where the sign depends on both the signature of the background and on the spacelike or timelike character of the curve. Indeed, the tensor $K_{\mu\nu}{}^\rho$ could be approached from a different point of view by taking (5.4) and (5.5) conjointly as its defining relations.

As in the Riemann tensor case, it is for many purposes useful to decompose the extrinsic imbedding curvature tensor into a *trace part*, K^ρ, that plays a role rather analogous to that of the Riemannian Ricci tensor, and a *trace free part*, $C_{\mu\nu}{}^{\rho}$, that plays a role rather analogous to that of the Weyl conformal tensor. It clearly follows from (4.16) that the *curvature vector* as given by

$$K^{\rho}=K^{\nu}{}_{\nu}{}^{\rho} \tag{5.7}$$

is the only independent trace part, and that it will have the surface orthogonality property

$$\eta^{\mu}{}_{\nu}K^{\nu}=0\,. \tag{5.8}$$

Just as the vanishing of the Ricci trace part of a background space Riemann tensor expresses the dynamic equations for the simple vacuum case of the Einstein gravitational theory as governed by the Hilbert action, so analogously [13,14] the vanishing of the extrinsic curvature vector as defined by (5.7) expresses the dynamic equations for the case of a simple cosmic string or membrane as governed by a Dirac–Goto–Nambu (surface measure) action.

When this trace is subtracted out one is left with what we shall refer to as the *conformation tensor*,

$$C_{\lambda\mu}{}^{\nu}=K_{\lambda\mu}{}^{\nu}-p^{-1}K^{\nu}\eta_{\lambda\mu}\,, \tag{5.9}$$

which is not only symmetric (by the Weingarten–Frobenius identity) but also trace free,

$$C_{[\lambda\mu]}{}^{\nu}=0\,,\qquad C_{\lambda\nu}{}^{\nu}=0\,, \tag{5.10}$$

and has the same mixed tangentiality and orthogonality properties as the full extrinsic curvature tensor,

$$\gamma_{\lambda}{}^{\sigma}C_{\sigma\mu}{}^{\nu}=C_{\mu\nu}{}^{\sigma}\eta_{\sigma}{}^{\nu}=0\,. \tag{5.11}$$

In terms of this decomposition, (5.5) can be written out as

$$\gamma^{\mu}{}_{\nu}\dot{u}^{\nu}=u^{\mu}u^{\nu}C_{\mu\nu}{}^{\rho}\mp p^{-1}K^{\rho}\,. \tag{5.12}$$

As well as being trace free, the conformation tensor that has just been introduced shares with the Weyl conformal tensor (to be discussed later on) the property of being *conformally invariant* when written with the appropriate mixture of raised and lowered indices. When the metric undergoes a general conformal transformation of the form

$$g_{\mu\nu}\mapsto \mathrm{e}^{-2\sigma}g_{\mu\nu}\,, \tag{5.13}$$

it is obvious that the mixed (projector) version $\eta_{\mu}{}^{\nu}$ of the first fundamental tensor and also its orthogonal complement $\gamma_{\mu}{}^{\nu}$ will be remain unaffected, i.e., we shall have

$$\eta^{\mu}{}_{\nu}\mapsto\eta^{\mu}{}_{\nu}\,,\qquad \gamma^{\mu}{}_{\nu}\mapsto\gamma^{\mu}{}_{\nu}\,, \tag{5.14}$$

and it is easy to verify for the second fundamental tensor that its trace free part will also be unaffected, i.e, we shall get

$$C_{\mu\nu}{}^{\rho} \mapsto C_{\mu\nu}{}^{\rho} , \tag{5.15}$$

whereas its trace, the intrinsic curvature vector, will suffer a non-trivial conformal adjustment proportional to the p-surface orthogonal projection of the derivative of the conformal factor, as given by

$$K_{\mu} \mapsto K_{\mu} + p\gamma_{\mu}{}^{\nu}\nabla_{\nu}\sigma . \tag{5.16}$$

The analogy between the conformation tensor and the Weyl conformal tensor (whose vanishing is the condition for conformal flatness in four or more dimensions) can be taken even further, since, as remarked in the introduction and shown below, in a conformally flat background the vanishing of $C_{\lambda\mu}{}^{\nu}$ is a sufficient condition for conformal flatness of the induced metric on the imbedded p-surface – as also for the vanishing of the outer curvature (i.e., for the existence of an external frame gauge for which the outer connection pseudo-tensor $\omega_{\lambda}{}^{\mu}{}_{\nu}=0$).

The best way to get a feeling for the working of the formalism that has just been set up is of course to see how it works out in particularly simple special cases. In the examples that follow, which treat the trivial and semi-trivial cases $p=1$ and $p=n-1$, respectively, the equations will be left unnumbered in order to advertise that their status is different from the numbered equations in the rest of the text, which are valid for generic values for the imbedding dimension p.

As a trivial illustration of the way the foregoing formalism works, the simplest special case is that of a (non-null) *curve* ($p=1$), for which the unit tangent vector, u^{μ} [as characterised by (5.6)] will be unique except for the sign of its orientation, which is irrelevant for the (quadratically dependent) results that follow. It will, however, be necessary to take account of the sign alternative in (5.6), which depends both on the temporal character of the curve and on the signature of the background metric. The first fundamental tensor, $\eta_{\mu\nu}$, its complement $\gamma_{\mu\nu}$ and the unit normalisation condition will evidently be given in this case by expressions of the form

$$\eta_{\mu\nu} = \mp u_{\mu}u_{\nu} , \qquad \gamma_{\mu\nu} = g_{\mu\nu} \pm u_{\mu}u_{\nu} ,$$

where the lower sign alternative applies to the case in which the background metric $g_{\mu\nu}$ is positive definite, whereas if it is of Lorentzian type either alternative is possible, the upper value corresponding to the case of a timelike curve when the standard MTW convention [17] is used. It follows that in terms of the "acceleration vector", as traditionally defined by (5.6), the second fundamental tensor will be given simply by

$$K_{\mu\nu}{}^{\rho} = \mp \eta_{\mu\nu}\dot{u}^{\rho} ,$$

which clearly means that the conformation tensor will merely vanish identically while the curvature vector will be proportional to the acceleration,

$$C_{\mu\nu}{}^{\rho}=0\,,\qquad K^{\rho}=\mp\dot{u}^{\rho}\,.$$

The opposite extreme case (to which most of the relevant literature has been restricted) is that of a (non-null) *hypersurface* ($p=n-1$), which is considerably simpler than the generic case but not as trivial as that of a curve. In this case there will be a unit normal, λ_ρ, that is well defined up to a choice of sign, which (unlike that in the tangent vector in the previous example), is not without significance for the formulae that follow, in which it will also be necessary to take account of a sign alternative depending both on the temporal character of the hypersurface and on the signature of the background metric. The first fundamental tensor, $\eta_{\mu\nu}$, its complement $\gamma_{\mu\nu}$ and the unit normalisation condition will evidently be given in this hypersurface case by

$$\eta_{\mu\nu}=g_{\mu\nu}\pm\lambda_\mu\lambda_\nu\,,\qquad \gamma_{\mu\nu}=\mp\lambda_\mu\lambda_\nu\,,\qquad \lambda_\rho\lambda^\rho=\mp 1\,,$$

where, as in the previous example, the lower sign alternative applies to the case in which the background metric $g_{\mu\nu}$ is positive definite, whereas if it is of Lorentzian type either alternative is possible: on the understanding that the MTW convention [17] is used, the upper alternative would apply to the important case [18] of a *spacelike initial value hypersurface*, whereas for a membrane world sheet in a four-dimensional spacetime it is the lower alternative that would apply. In the case of any such hypersurface, one can construct a *two-index* symmetric imbedding curvature tensor that is well defined, modulo the inevitable sign ambiguity in the choice of orientation of the normal λ_ρ, by the formula

$$K_{\mu\nu}=K_{\mu\nu}{}^{\rho}\lambda_\rho\,.$$

The restriction of this (background) tensor to the hypersurface [to which it is already automatically hypersurface tangential by (4.16)] is what is usually known as the *second fundamental form* [16]. The full second fundamental tensor [whose definition (5.2), unlike that of the second fundamental form, involves no sign ambiguities whatsoever] will then be expressible as

$$K_{\mu\nu}{}^{\rho}=\mp K_{\mu\nu}\lambda^\rho\,,$$

while the curvature vector will be expressible in terms of the scalar K that is identifiable as the trace of the second fundamental form by the formula

$$K_\rho=\mp K\lambda_\rho\,,\qquad K=K_\nu{}^{\nu}=K^\rho\lambda_\rho\,.$$

Last, but as far as its interest for the present work is concerned not least, the conformation tensor of a hypersurface in n dimensions will be given by

$$C_{\mu\nu}{}^{\rho}=\mp C_{\mu\nu}\lambda^\rho\,,\qquad C_{\mu\nu}=K_{\mu\nu}-\frac{1}{n-1}K\eta_{\mu\nu}\,.$$

Although obviously trace free by construction, the (hypersurface tangential) tensor $C_{\mu\nu}$ that (modulo a choice of sign depending on the orientation of the normal) is defined in this way, and whose restriction to the hypersurface might naturally be termed the "conformation form", is *not* strictly invariant (unlike the full conformation tensor $C_{\mu\nu}{}^{\rho}$) under the conformal transformation (5.13), whose effect will be expressible by

$$C_{\mu\nu} \mapsto \mathrm{e}^{\sigma} C_{\mu\nu}\,, \qquad K \mapsto \mathrm{e}^{\sigma}[K + (n-1)\lambda^{\rho} \nabla_{\rho}\sigma]\,.$$

6. The third fundamental tensor and the Codazzi equation

Having seen how, by eq. (5.3), complete first-order differential information about the imbedding is contained in the [by (5.2) manifestly gauge invariant] second fundamental tensor $K_{\lambda\mu}{}^{\nu}$, it is natural to go on in an analogous way to introduce what we shall refer to as the *third fundamental tensor* $\Xi_{\kappa\lambda\mu}{}^{\nu}$, constructed as the correspondingly projected derivative of the second fundamental tensor, namely

$$\Xi_{\kappa\lambda\mu}{}^{\nu} = \eta_{\lambda}{}^{\rho}\eta_{\mu}{}^{\sigma}\gamma_{\tau}{}^{\nu}\bar{\nabla}_{\kappa}K_{\rho\sigma}{}^{\tau}\,. \tag{6.1}$$

In conjunction with the second fundamental form, this new (manifestly frame gauge independent) tensor does, as required, contain second-order differential information about the imbedding: the derivative components that are projected out in the defining construction can be recovered by a higher-order analogue of (5.3), which takes the form

$$\bar{\nabla}_{\kappa}K_{\lambda\mu}{}^{\nu} = \Xi_{\kappa\lambda\mu}{}^{\nu} + 2K_{\kappa}{}^{\sigma}{}_{(\lambda}K_{\mu)\sigma}{}^{\nu} - K_{\kappa}{}^{\nu}{}_{\sigma}K_{\lambda\mu}{}^{\sigma}\,. \tag{6.2}$$

It is obvious from the way it has been constructed that this third fundamental tensor has the symmetry property

$$\Xi_{\kappa[\lambda\mu]}{}^{\nu} = 0 \tag{6.3}$$

as well as the properties of tangentiality (to the imbedding) of its first three indices and orthogonality of the last, i.e.,

$$\gamma_{\kappa}{}^{\sigma}\Xi_{\sigma\lambda\mu}{}^{\nu} = \gamma_{\lambda}{}^{\sigma}\Xi_{\kappa\sigma\mu}{}^{\nu} = \Xi_{\kappa\lambda\mu}{}^{\sigma}\eta_{\sigma}{}^{\nu} = 0\,. \tag{6.4}$$

By comparing its frame component expression with that of the corresponding projection of the background Riemann curvature tensor $B_{\kappa\lambda}{}^{\mu}{}_{\nu}$ as given in the next section, we obtain a version of what Eisenhart [1] recognised as the natural generalisation of the classical Codazzi equation for hypersurfaces, in the form

$$2\Xi_{[\kappa\lambda]\mu}{}^{\nu} = \eta_{\kappa}{}^{\rho}\eta_{\lambda}{}^{\sigma}\eta_{\mu}{}^{\tau}\gamma_{\upsilon}{}^{\nu}B_{\rho\sigma}{}^{\upsilon}{}_{\tau}\,. \tag{6.5}$$

In cases for which the background geometry has *constant curvature* in the sense exemplified by De Sitter space, meaning that the background Weyl tensor and the trace free part of the background Ricci tensor both vanish so that the background Ricci curvature is fully determined in terms of a constant background Ricci scalar B by an expression of the form

$$B_{\kappa\lambda}{}^{\mu\nu}=\frac{2}{n(n-1)}g_{[\kappa}{}^{[\mu}g_{\lambda]}{}^{\nu]}B\,, \tag{6.6}$$

it can be seen that the generalised Codazzi equation (6.5) will reduce to a form that is interpretable just as a condition of complete symmetry among the tangential indices of the third fundamental tensor, as expressed by

$$\Xi_{\kappa\lambda\mu}{}^{\nu}=\Xi_{(\kappa\lambda\mu)}{}^{\nu}\,. \tag{6.7}$$

In particular, this strong symmetry property of the third fundamental tensor will be valid for any imbedding in a background that is *flat*.

7. Background curvature formulae

We have introduced the symbol B for the n-dimensional *background* curvature (which in a more unwieldy but systematic nomenclature [8] would be indicated by ${}^{(n)}R$) in order to be able to reserve the usual symbol R for the *internal* Riemann curvature of the p-dimensional imbedded surface (which in the dimensionally systematic nomenclature [8] would be indicated by ${}^{(p)}R$) that will be worked out in the next section.

This background curvature tensor can be specified by the succinct expression

$$B_{\kappa\lambda\mu\Psi}=2\nabla_{[\kappa}\beta_{\lambda]\mu\Psi}\,, \tag{7.1}$$

or equivalently, by the more familiar albeit slightly longer Cartan type formula

$$B_{\kappa\lambda}{}^{\Phi}{}_{\Psi}=2\nabla_{[\kappa}\beta_{\lambda]}{}^{\Phi}{}_{\Psi}+2\beta_{[\kappa}{}^{\Phi\Theta}\beta_{\lambda]\Theta\Psi}\,, \tag{7.2}$$

the vanishing of this field being the condition that is both necessary and (locally) sufficient for it to be possible to choose the frame vectors $\theta_{\Lambda}{}^{\mu}$ in such a way as to get the frame section coefficients $\beta_{\Theta}{}^{\Phi}{}_{\Psi}$ to vanish.

Starting from (7.2), one can easily obtain the more complicated pure frame component version

$$B_{\Lambda\Theta}{}^{\Phi}{}_{\Psi}=2\nabla_{[\Lambda}\beta_{\Theta]}{}^{\Phi}{}_{\Psi}-2\beta_{[\Lambda}{}^{\Sigma}{}_{\Theta]}\beta_{\Sigma}{}^{\Phi}{}_{\Psi}-2\beta_{[\Lambda}{}^{\Phi\Sigma}\beta_{\Theta]\Phi\Sigma}\,, \tag{7.3}$$

which will be needed in the next section, and one obtains the corresponding purely tensorial and, as is well known, strictly (albeit not manifestly) frame independent version needed for (6.5) as

$$B_{\kappa\lambda}{}^{\mu}{}_{\nu}=2\nabla_{[\kappa}\beta_{\lambda]}{}^{\mu}{}_{\nu}+2\beta_{[\kappa}{}^{\mu\sigma}\beta_{\lambda]\sigma\nu}\,. \tag{7.4}$$

It is of course useful for many purposes to separate out the Ricci contractions

$$B_{\mu\nu}=B_{\mu\sigma\nu}{}^{\sigma}\,, \qquad B=B_{\sigma}{}^{\sigma}\,, \tag{7.5}$$

which (following Schouten [3]) one can conveniently combine in the form

$$\tilde{B}_{\mu\nu}=B_{\mu\nu}-\frac{1}{2(n-1)}Bg_{\mu\nu} \tag{7.6}$$

(where it is to be recalled that n is the dimension of the background spacetime under consideration). This makes it possible – assuming that the background spacetime dimension is at least three – to specify the background *Weyl tensor* as

$$W_{\kappa\lambda}{}^{\mu\nu}=B_{\kappa\lambda}{}^{\mu\nu}-\frac{4}{n-2}g_{[\kappa}{}^{[\mu}\tilde{B}_{\lambda]}{}^{\nu]}\,. \tag{7.7}$$

From the usual Riemann and Ricci symmetry properties

$$B_{\kappa\lambda\mu\nu}=B_{[\kappa\lambda][\mu\nu]}\,, \qquad B_{[\kappa\lambda\mu]\nu}=0\,, \tag{7.8}$$

$$B_{\kappa\lambda\mu\nu}=B_{\mu\nu\kappa\lambda}\,, \qquad B_{\mu\nu}=B_{\nu\mu}\,, \tag{7.9}$$

the Weyl tensor inherits the corresponding symmetry properties

$$W_{\kappa\lambda\mu\nu}=W_{[\kappa\lambda][\mu\nu]}=W_{\mu\nu\kappa\lambda}\,, \qquad W_{[\kappa\lambda\mu]\nu}=0\,, \tag{7.10}$$

and it has the further property of being entirely trace free,

$$W_{\mu\sigma}{}^{\nu\sigma}=0\,. \tag{7.11}$$

It is well known [3] that the vanishing of the Weyl tensor is a necessary condition for conformal flatness – again assuming that the spacetime dimension n is at least three. (If the spacetime dimension were only two, conformal flatness would hold automatically while the Weyl tensor would be undefined.) The condition that $W_{\kappa\lambda\mu\nu}$ should vanish is not only necessary but also sufficient for conformal flatness provided the background dimension satisfies $n\geqslant 4$.

In the special case for which $n=3$ the Weyl tensor is identically zero, whereas conformal flatness is still a non-trivial restriction, a sufficient as well as necessary condition in this case being the vanishing of the higher derivative tensor [3]

$$\tilde{B}_{\mu\nu\rho}=\nabla_{[\mu}\tilde{B}_{\nu]\rho}\,. \tag{7.12}$$

(In higher dimensions, $n\geqslant 3$, the vanishing of this tensor $\tilde{B}_{\mu\nu\rho}$ is a necessary consequence of the vanishing of the Weyl tensor and hence remains always necessary for conformal flatness, even though it ceases to be sufficient.)

At the higher differential order involved in (7.12) it is to be recalled that we shall always have the Bianchi integrability condition

$$\nabla_{[\kappa}B_{\lambda\mu]}{}^{\sigma}{}_{\tau}=0 \tag{7.13}$$

holding as an identity for an arbitrary background metric.

8. Internal curvature and the Gauss equation

By a procedure analogous to that described for the background spacetime in the previous section, we can now go on to construct the corresponding *internal curvature tensor* with frame components $R_{AB}{}^{C}{}_{D}$, say, whose vanishing will be the necessary and (locally) sufficient condition for it to be possible to choose the tangential frame vectors $\iota_A{}^{\mu}$ in the imbedded p-surface in such a way as to get the corresponding internal connection coefficients $\rho_A{}^{B}{}_{C}$ to vanish.

In terms of these purely internal rotation coefficients $\rho_A{}^{B}{}_{C}$ as given by (4.7) (or as equivalently obtainable purely in terms of the induced p-dimensional geometry within the surface), the required frame components of the internal Riemann curvature of the imbedded p-surface will evidently be given by the corresponding p-dimensional analogue of (7.3), namely

$$R_{AB}{}^{C}{}_{D}=2\nabla_{[A}\rho_{B]}{}^{C}{}_{D}+2\rho_{[A}{}^{CE}\rho_{B]ED}-2\rho_{[A}{}^{E}{}_{B]}\rho_{E}{}^{C}{}_{D}\,. \tag{8.1}$$

This expression could be used to obtain the components of the Riemannian curvature of the induced metric with respect to a system of internal coordinates within the imbedded p-surface by expressing the tangential frame vectors in terms of such a system, but we shall not do this here in view of our resolution to work only in terms of a single system of background coordinates that are specified in advance independently of any particular imbedding.

When projected back into terms of the corresponding background coordinate components according to the prescription $R_{\kappa\lambda}{}^{\mu}{}_{\nu}=\iota^{A}{}_{\kappa}\iota^{B}{}_{\lambda}\iota_{C}{}^{\mu}\iota^{D}{}_{\nu}R_{AB}{}^{C}{}_{D}$, the resulting spacetime version of the inner curvature is found to be given directly by the (formally) tensorial expression [the analogue of (7.4)] in the form

$$R_{\kappa\lambda}{}^{\mu}{}_{\nu}=2\eta_{\sigma}{}^{\mu}\eta_{\nu}{}^{\tau}\eta_{[\lambda}{}^{\pi}\overline{\nabla}_{\kappa]}\rho_{\pi}{}^{\sigma}{}_{\tau}+2\rho_{[\kappa}{}^{\mu\pi}\rho_{\lambda]\pi\nu} \tag{8.2}$$

in terms of the pseudo-tensorial quantities defined by (4.12) and the tangential covariant differentiation operator defined by (5.1). It will not only have the usual Riemann symmetry properties

$$R_{\kappa\lambda\mu\nu}=R_{[\kappa\lambda][\mu\nu]}=R_{\mu\nu\kappa\lambda}\,,\qquad R_{[\kappa\lambda\mu]\nu}=0\,, \tag{8.3}$$

but will evidently also, by construction, be characterised by the property of being purely tangential to the p-surface, like the fundamental tensor $\eta_{\mu}{}^{\nu}$ itself, i.e., it will vanish under the action of any orthogonal projection,

$$\gamma_{\kappa}{}^{\sigma}R_{\sigma\lambda\mu\nu}=0\,. \tag{8.4}$$

Analogous remarks naturally apply to the Ricci contractions

$$R_{\mu\nu}=R_{\mu\sigma\nu}{}^{\sigma}\,,\qquad R=R_{\sigma}{}^{\sigma}\,, \tag{8.5}$$

for which we shall have

$$R_{[\mu\nu]}=0\,,\qquad \gamma_{\mu}{}^{\sigma}R_{\sigma\nu}=0\,. \tag{8.6}$$

Continuing along the lines suggested by the standard procedures described for the background spacetime curvature in the preceding section, we can go on to define the internal analogue of Schouten's trace adjusted Ricci tensor (7.6) as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2(p-1)} R\eta_{\mu\nu} , \tag{8.7}$$

which will of course satisfy

$$\tilde{R}_{[\mu\nu]} = 0 , \qquad \gamma_\mu{}^\sigma \tilde{R}_{\sigma\nu} = 0 . \tag{8.8}$$

This will identically vanish in the special case (which applies to "string" models) of an imbedded surface that is two dimensional, $p=2$, for which we shall have

$$R_{\kappa\lambda}{}^{\mu\nu} = R\eta_{[\kappa}{}^{[\mu}\eta_{\lambda]}{}^{\nu]} , \qquad \tilde{R}_{\mu\nu} = 0 \quad . \tag{8.9}$$

Otherwise, i.e. for imbedded surfaces of higher dimension, $p \geqslant 3$, the internal curvature tensor will be decomposable in the form

$$R_{\kappa\lambda}{}^{\mu\nu} = C_{\kappa\lambda}{}^{\mu\nu} + \frac{4}{p-2} \eta_{[\kappa}{}^{[\mu}\tilde{R}_{\lambda]}{}^{\nu]} , \tag{8.10}$$

where the quantity $C_{\kappa\lambda\mu\nu}$ that thus makes its appearance is what we shall refer to as the *internal conformal curvature tensor*, its role being evidently analogous to that of the background Weyl tensor that was introduced in (7.7). It is evident that, as well as inheriting algebraic symmetry properties of the same form as those expressed by (8.3) for the full internal curvature tensor, it will in addition not only have the same tracelessness property as that expressed by (7.11) for the ordinary background Weyl tensor, but will also have the property of tangentiality, i.e, vanishing projections orthogonal to the p-surface:

$$C_{\mu\sigma}{}^{\nu\sigma} = 0 , \qquad \gamma_\kappa{}^\sigma C_{\sigma\lambda}{}^{\mu\nu} = 0 . \tag{8.11}$$

It is obvious (from the point of view of the frame component approach) that the theorem [3] that was quoted in the previous section can be taken over from the background spacetime geometry to the internal geometry of the imbedded p-surface, the implication being that the vanishing of the internal conformal curvature tensor $C_{\kappa\lambda\mu\nu}$ is both necessary and sufficient for the internal geometry of the p-surface to be conformally flat provided $p \geqslant 4$. In the special case $p=3$ (that is relevant to membrane dynamics in ordinary spacetime) the internal conformal curvature tensor will vanish automatically, and as a sufficient condition for conformal flatness of the internal geometry of the imbedded three-surface it will then be necessary to require the vanishing of the analogue of the tensor introduced in (6.12), i.e., of the tangentially projected higher derivative tensor that is specifiable as

$$\tilde{R}_{\lambda\mu\nu} = \eta_\nu{}^\tau \eta_{[\mu}{}^\sigma \bar{\nabla}_{\lambda]} \tilde{R}_{\sigma\tau} . \tag{8.12}$$

At the higher differential order involved in (8.12) the internal curvature will of course be subject to a generalised Bianchi integrability condition that is the analogue of the background Bianchi identity (7.13). The relevant Bianchi identity for the internal curvature of an arbitrary spacelike or timelike imbedded p-surface in a general Riemannian or pseudo-Riemannian background is obtainable in the form

$$\eta_{[\kappa}{}^{\nu}\eta_{\lambda}{}^{\rho}\bar{\nabla}_{\mu]}R_{\nu\rho}{}^{\sigma\tau}=2R_{[\kappa\lambda}{}^{\nu[\tau}K_{\mu]\nu}{}^{\sigma]}\,, \tag{8.13}$$

whose fully tangential projection into the p-surface is just the corresponding ordinary p-dimensional Bianchi identity [the right hand side of (8.13) being obviously taken out by the tangential projection operation].

In the absence of information about the background curvature tensor, the internal curvature tensor $R_{\kappa\lambda}{}^{\mu}{}_{\nu}$ as introduced in the present section can be considered to be algebraically independent of the second fundamental tensor $K_{\kappa\lambda}{}^{\mu}$ that was presented in sections 4 and 5. However, when the background curvature tensor $B_{\kappa\lambda}{}^{\mu}{}_{\nu}$ is known then the obvious possibility of identifying the quantities appearing on the right of (8.1) as a subset of those appearing on the right of (7.3) can be used to evaluate the internal Riemann tensor frame components and hence (by contraction with the relevant tangential frame vectors $\iota_A{}^{\mu}$) to obtain the corresponding spacetime coordinate components in the form

$$R_{\kappa\lambda}{}^{\mu}{}_{\nu}=2K_{[\kappa}{}^{\mu\sigma}K_{\lambda]\nu\sigma}+\eta_{\kappa}{}^{\rho}\eta_{\lambda}{}^{\sigma}\eta_{\tau}{}^{\mu}\eta_{\nu}{}^{\upsilon}B_{\rho\sigma}{}^{\tau}{}_{\upsilon}\,. \tag{8.14}$$

This expression is interpretable [1] as a generalisation of the historic Gauss equation for an imbedded hypersurface, and has the advantage over (8.2) of making the *strictly* tensorial (frame independent) nature of the internal curvature more directly apparent, as a manifest consequence of the strictly tensorial (frame independent) nature of the background curvature tensor and of the first and second fundamental tensors as previously introduced.

The simplest non-trivial applications of the concept of inner curvature are of course those, including the physically important case of string models, for which the imbedding has dimension $p=2$ with the implication that the inner rotation group is Abelian so that the second term in (8.2) will drop out. When $p=2$ the imbedding surface will be characterised by an antisymmetric unit tangent element tensor given by

$$\mathcal{E}^{\mu\nu}=\mathcal{E}^{[\mu\nu]}=\mathcal{E}^{AB}\iota_A{}^{\mu}\iota_B{}^{\nu}$$

(where $\mathcal{E}^{AB}$ are the constant components of the standard two-dimensional flat space alternating tensor), which can be considered as a square root of the first fundamental tensor, which will be given simply by

$$\eta^{\mu}{}_{\nu}=\pm\mathcal{E}^{\mu}{}_{\sigma}\mathcal{E}^{\sigma}{}_{\nu}\,,$$

where the upper (positive) sign applies to the case of a timelike world sheet and the lower (negative) sign to the case when the imbedded two-surface has an induced metric that is positive definite. Since the analogue of (5.3) governing the tangential derivative of this surface element tensor will be expressible in terms of the second fundamental tensor by

$$\bar{\nabla}_\sigma \mathcal{E}^{\mu\nu} = 2K_{\sigma\tau}{}^{[\nu}\mathcal{E}^{\mu]\tau} ,$$

whose fully surface tangential (as also its fully surface orthogonal) projection can be seen to vanish identically, it follows that the contraction of (8.2) with the surface element tensor gives an identity of the simple form

$$R\mathcal{E}_{\kappa\lambda} = R_{\kappa\lambda}{}^{\mu}{}_{\nu}\mathcal{E}^{\nu}{}_{\mu} = 2\eta_{[\lambda}{}^{\sigma}\bar{\nabla}_{\kappa]}\rho_\sigma ,$$

with

$$\rho_\sigma = \rho_\sigma{}^{\mu}{}_{\nu}\mathcal{E}^{\nu}{}_{\mu}, \qquad \gamma_\sigma{}^{\tau}\rho_\tau = 0 ,$$

which shows (see appendices) that the restriction of the two-form $R\mathcal{E}_{\mu\nu}$ to the imbedding two-surface is the *exterior derivative* of a (locally defined, frame gauge dependent) one-form (covector) ρ_μ in the surface, this property being what gives rise [18,19] to the topological invariance of the corresponding generalised Gauss–Bonnet type integral that is obtainable by integration of the curvature scalar R over the entire imbedding two-surface subject to appropriate boundary conditions if it is non-compact (as will normally be the case for the timelike world sheet of a string model).

9. The outer curvature tensor

The construction of an appropriate curvature tensor, whose vanishing is necessary and (locally) sufficient for the frame in question to be adjustable so as to eliminate the corresponding connection coefficients, can now be carried out for the *outer* (surface orthogonal) frame $\lambda_R{}^\mu$ by a procedure very similar to that employed in the previous section for the inner (tangential) frame. This is done by applying, with respect to the relevant $(n-p)$-dimensional *outer* rotation group, the standard gauge theoretical principles that were first systematically developed at the time of Cartan, and that have become widely known [12,13] since the introduction of Yang–Mills theory, but that were not familiar to classical geometers of the nineteenth and early twentieth century, up to and including such an authority as Eisenhart [1]. One thus obtains the Cartan frame components of the required *outer curvature tensor* as

$$\Omega_{AB}{}^{R}{}_{S} = 2\nabla_{[A}\omega_{B]}{}^{R}{}_{S} + 2\omega_{[A}{}^{RT}\omega_{B]TS} - 2\rho_{[A}{}^{C}{}_{B]}\omega_{C}{}^{R}{}_{S} , \qquad (9.1)$$

recalling that the early Latin indices (A, B, C) refer to the tangential frame within the p-surface under consideration, while the late Latin indices (R, S, T) refer to the outer frame whose $(n-p)$-dimensional rotations [as characterised by the preservation of the diagonal constant product matrix γ_{RS} given by (4.10)] constitute the gauge group in question.

The first two terms on the right of (9.1) are of the kind that are familiar from flat space Yang–Mills theory, while the remaining last term on the right could be made to drop out by using a flat connection if the inner curvature $R_{AB}{}^{C}{}_{D}$ of the p-surface were zero. The corresponding term would also drop out quite generally if we went over to the equivalent formula as expressed in terms of partial differentiation with respect to an internal coordinate system (such as we have refrained from introducing in the present work) on the p-surface, but the extra term is needed in the version (9.1) because it uses frame oriented covariant differentiation rather than the Cartan procedure of antisymmetrised partial differentiation of differential forms. There is, however, no need to introduce an internal coordinate system to get rid of the extra term; it will also drop out of its own accord when we go over from the frame version (9.1) [the outer analogue of (8.1)] to the background coordinate version [the outer analogue of (8.2)] for the tensorial form $\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}=\iota^{A}{}_{\kappa}\iota^{B}{}_{\lambda}\Omega_{AB}{}^{R}{}_{S}\lambda_{R}{}^{\mu}\lambda^{S}{}_{\nu}$ of the outer curvature. This spacetime tensorial version will be given, in terms of the tangential covariant differentiation operator $\bar{\nabla}_{\mu}$ introduced in (5.1), by the comparatively simple formula

$$\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}=2\gamma_{\sigma}{}^{\mu}\gamma_{\nu}{}^{\tau}\eta_{[\lambda}{}^{\pi}\bar{\nabla}_{\kappa]}\omega_{\pi}{}^{\sigma}{}_{\tau}+2\omega_{[\kappa}{}^{\mu\pi}\omega_{\lambda]\pi\nu}\,, \tag{9.2}$$

which, like (8.2), gives a frame independent result despite the frame dependence of the separate contributions on the right hand side as defined by (4.12). Unlike the inner curvature tensor, which has the full set of Riemann symmetries (8.3), the outer curvature tensor only has the more restricted set of symmetries that are expressed by

$$\Omega_{\mu\nu\rho\sigma}=\Omega_{[\mu\nu][\rho\sigma]}\,, \tag{9.3}$$

i.e., is is antisymmetric on the first and last pair of indices taken separately. It also differs from its inner analogue in that, whereas it is of course still tangential on the first pair of indices, on the other hand it is orthogonal on the last pair, i.e.,

$$\gamma_{\mu}{}^{\tau}\Omega_{\tau\nu\rho\sigma}=0\,,\qquad \Omega_{\mu\nu\rho\tau}\eta^{\tau}{}_{\sigma}=0\,. \tag{9.4}$$

It obviously follows that the outer curvature tensor is purely Weyl-like in the sense that the analogue of the Ricci contraction (8.5) is identically zero,

$$\Omega_{\sigma\lambda}{}^{\sigma}{}_{\nu}=0\,. \tag{9.5}$$

On the basis of the well-known generalised Bianchi identity that holds for the curvature of any kind of differential connection [19], one can verify in this case that the analogue for the outer curvature of the identity (8.13) is the higher differential identity that is expressible as

$$\eta_{[\kappa}{}^{\nu}\eta_{\lambda}{}^{\rho}\bar{\nabla}_{\mu]}\Omega_{\nu\rho}{}^{\sigma\tau}=2\Omega_{[\kappa\lambda}{}^{\nu[\sigma}K_{\mu]}{}^{\tau]}{}_{\nu}\,. \tag{9.6}$$

Starting from our original frame component expression (9.1), the formalism that has been developed now makes it straightforward to work out the "third" (Voss–Ricci–Walker–Schouten) relation of the trio referred to in the introduction, i.e., the outer analogue of the generalised Gauss relation (8.14) between inner curvature and the relevant projection of the background curvature. As in the previous case one obtains a term that is quadratically dependent on the extrinsic imbedding curvature tensor, $K_{\mu\nu}{}^{\rho}$. The outer curvature is thus finally evaluated, in a form that [unlike the previous expression (9.2)] makes its strictly tensorial (frame independent) nature obvious, as

$$\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}=2K_{[\kappa}{}^{\sigma\mu}K_{\lambda]\sigma\nu}+\eta_{\kappa}{}^{\rho}\eta_{\lambda}{}^{\sigma}\gamma_{\tau}{}^{\mu}\gamma_{\nu}{}^{\upsilon}B_{\rho\sigma}{}^{\tau}{}_{\upsilon}\,. \tag{9.7}$$

The symmetry conditions (9.3) clearly imply that the outer curvature can be non-zero only if neither the surface dimension p nor the codimension $(n-p)$ are less than two, a condition that rules out both the case of simple one-dimensional curves and the case of hypersurfaces. In a four-dimensional background spacetime, the only case in which non-trivial outer curvature can occur is that for which the imbedded surface is two dimensional (as is the case for the world sheets of strings), and even in this case the outer curvature can only be of the Abelian kind (i.e. of Maxwellian rather than general Yang–Mills type) since in order for the outer frame gauge group to be non-Abelian it is clearly necessary for the complementary dimension $(n-p)$ to be at least 3, which means that non-Abelian gauge curvature can only occur if the background dimension n is at least 5 [as is the case in Kaluza–Klein models, but not in ordinary spacetime, which is doubtless one of the reasons why relation (9.7) would seem to have been previously overlooked in the literature of theoretical physics].

In view of (9.5) it is evident in advance that the net result given by (9.7) must be identically trace free. However, the most noteworthy feature of this outer curvature equation is the separate cancelling out of the separate trace parts (namely the extrinsic imbedding curvature K^{ρ} and the background Ricci tensor $B_{\mu\nu}$) in the source contributions on the right, so that in a spacetime background of dimension $n\geqslant 3$ the final result is expressible purely in terms of the trace free conformation tensor $C_{\mu\nu}{}^{\rho}$ given by (5.9), and of the Weyl conformal tensor $W_{\kappa\lambda}{}^{\mu}{}_{\nu}$ of the background as given by (7.7), in the form

$$\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}=2C_{\sigma[\kappa}{}^{\mu}C_{\lambda]}{}^{\sigma}{}_{\nu}+\eta_{\kappa}{}^{\rho}\eta_{\lambda}{}^{\sigma}\gamma_{\tau}{}^{\mu}\gamma_{\nu}{}^{\upsilon}W_{\rho\sigma}{}^{\tau}{}_{\upsilon}\,, \tag{9.8}$$

which is to be considered as the definitive version of the "third" equation in the generic case. Since it is apparent that each of the two separate terms on the right is separately conformally invariant, we can deduce as an immediate corollary that the outer curvature tensor $\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}$ is itself conformally invariant.

The very convenient and widely used Walker [6] generalisation to an arbitrarily accelerated curve of what in the special case of a geodesic are traditionally known as "Fermi coordinates" is generated by an external frame that is propagated according to a rule expressible as the condition that the corresponding outer rotation coefficients should vanish. This Walker (or, as it is commonly called, "Fermi–Walker") propagation rule is thus a special case of a more general propagation ansatz expressible in the notation used here as the requirement that the outer gauge connection coefficients $\omega_R{}^S{}_T$ should all be zero. Unfortunately (from the point of view of many applications [20,21] for which the corresponding generalised Walker coordinate system would be useful) the possibility of imposing such "outer-flat" propagation is limited to cases in which the corresponding outer curvature tensor $\Omega_{\kappa\lambda}{}^\mu{}_\nu$ is zero, which, as remarked in the previous paragraph, is guaranteed in advance for a curve (the case originally considered by Walker) or a hypersurface, but not for imbedded surfaces of intermediate dimension, such as the case of a string in four dimensions or a membrane in a higher-dimensional background.

In any background that is conformally flat (at least to a sufficiently good approximation over the length scales under consideration, as will very often be the case), and in any three-dimensional spacetime background whatsoever, the Weyl tensor $W_{\kappa\lambda}{}^\mu{}_\nu$ will vanish so that it will follow from (9.8) that a sufficient condition for the generalised Walker "outer-flat" propagation condition to be imposable is the vanishing of the conformation tensor as defined by (5.9). Although it will thus be sufficient, the vanishing of this conformation tensor is clearly not necessary for outer flatness, since, while it vanishes identically in the case of a curve, $C_{\mu\nu}{}^\rho$ can evidently be non-zero in the trivially outer-flat case of a hypersurface (see the final paragraph of section 5).

The simplest non-trivial applications of the concept of outer curvature are those, including the physically important case of string models in a background space of four (but not more) dimensions, for which the imbedding has codimension $n-p=2$ with the implication that the outer rotation group is Abelian so that the second term in (9.2) will drop out. When $n-p=2$ the (Hodge type) dual to the p-index antisymmetric surface measure tensor of the imbedding will be a two-index antisymmetric tensor orthogonal to the surface given by

$$\mathcal{E}_{\bullet\mu\nu}=\mathcal{E}_{\bullet[\mu\nu]}=\mathcal{E}_{\bullet RS}\lambda^R{}_\mu\lambda^S{}_\nu$$

(where $\mathcal{E}_{\bullet RS}$ are the constant components of a standard two-dimensional flat space alternating tensor), which can be considered as a square root of the orthogonal complement of the first fundamental tensor, which will be given simply by

$$\gamma^\mu{}_\nu=\mp\,\mathcal{E}_\bullet{}^\mu{}_\sigma\mathcal{E}_\bullet{}^\sigma{}_\nu\,,$$

where the upper (negative) sign applies to the case of a timelike world sheet. Since the analogue of (5.3) governing the tangential derivative of this surface element tensor will be expressible in terms of the second fundamental tensor by

$$\bar{\nabla}_\sigma \mathcal{E}_{*\mu\nu} = 2K_{\sigma[\mu}{}^\tau \mathcal{E}_{*\nu]\tau} ,$$

whose fully surface orthogonal (as also its fully surface tangential) projection can be seen to vanish identically, it follows that the contraction of (9.2) with the surface element tensor gives an identity of the simple form

$$\Omega_{\mu\nu} = \Omega_{\kappa\lambda}{}^\mu{}_\nu \mathcal{E}_*{}^\nu{}_\mu = 2\eta_{[\lambda}{}^\sigma \bar{\nabla}_{\kappa]} \omega_\sigma ,$$

with

$$\omega_\sigma = \omega_\sigma{}^\mu{}_\nu \mathcal{E}_*{}^\nu{}_\mu , \qquad \gamma_\sigma{}^\tau \omega_\tau = 0 ,$$

which specifies a (geometrically well-defined) two-form $\Omega_{\mu\nu}$ ($=\Omega_{[\mu\nu]}$) with the property that (as in the analogous case of the inner curvature of an imbedded two-surface as discussed at the end of the previous section) its restriction to the $(n-2)$-surface of the imbedding is the exterior derivative (see appendices) of a (locally defined, frame gauge dependent) one-form ω_μ in the surface.

In the special case of a four-dimensional background, the outer curvature of a surface of codimension (and hence also dimension) two will be fully determined by a single (pseudo-)scalar invariant, Ω say, in terms of which the surface-closed curvature two-form introduced in the preceding paragraph will be expressible as

$$\Omega_{\mu\nu} = \Omega \mathcal{E}_{\mu\nu} , \qquad 2\Omega = \epsilon^{\mu\nu}{}_{\rho\sigma} \Omega_{\mu\nu}{}^{\rho\sigma}$$

(where $\epsilon_{\mu\nu\rho\sigma}$ denotes the standard fully antisymmetric four-volume measure tensor of the background, in terms of which we shall in this case have $2\mathcal{E}_{*\mu\nu} = \epsilon_{\mu\nu}{}^{\rho\sigma} \mathcal{E}_{\rho\sigma}$). In this case, the integration of this two-form over the entire imbedding two-surface subject to appropriate boundary conditions if it is non-compact (as will normally be the case for a timelike world sheet) will give rise to a topological invariant that can be considered as an "outer" analogue of an ordinary "inner" Gauss–Bonnet type invariant of the kind mentioned at the end of the previous section. In particular this applies for an ordinary spacetime background to the physically interesting case of a string world sheet, and also to the case of a spacelike two-surface, which was studied by Penrose [22,11], who showed how a spinor approach leads naturally to the construction of a single *complex* curvature invariant that effectively combines the two independent (real) inner (scalar) and outer (pseudo-scalar) curvature invariants R and Ω (the corresponding combination of their surface integrals thus giving a complex generalisation of the ordinary real Gauss–Bonnet type global topological invariant).

10. The internal curvature in a conformally flat background

The conclusion of the preceding section is an illustration of the critically significant role of the conformation tensor $C_{\mu\nu}{}^\rho$ of an imbedding when the background is conformally flat, which suggests that it will be of interest to make a closer examination of its role with respect to the *inner* curvature, $R_{\kappa\lambda}{}^{\mu}{}_{\nu}$, and more particularly of its tensorially irreducible parts, in this conformally flat case, i.e., when the conditions

$$W_{\kappa\lambda}{}^{\mu}{}_{\nu}=0\,, \tag{10.1}$$

$$\tilde{B}_{\lambda\mu\nu}=0\,, \tag{10.2}$$

are both satisfied (the second of these conditions being a consequence of the first except when the background dimension is $n=3$, in which case the first is merely an identity [3]). This restriction is of course compatible with all the most common kinds of application, in which the background is taken to be not just conformally flat, but flat in the strong sense, which is justifiable at least as a very good approximation in a very wide range of circumstances in which the characteristic length scales of the imbedding will be small compared with those of the background curvature if any. Although it is unnecessary for such cases, we shall nevertheless retain allowance for the possibility of a non-zero background Ricci tensor $B_{\mu\nu}$ in the formulae that follow since the extra complication involved thereby is only very moderate (compared with what would result if allowance for a non-zero background Weyl tensor were also included).

Subject to (10.1) and leaving aside the trivial (always locally conformally flat) case of a two-dimensional background, the generalised Gauss relation (8.14) reduces to the form

$$R_{\kappa\lambda}{}^{\mu}{}_{\nu}=2K_{[\kappa}{}^{\mu\sigma}K_{\lambda]\nu\sigma} + \frac{2}{n-2}\left(\eta_{[\kappa}{}^{\mu}\eta_{\lambda]}{}^{\rho}\eta_{\nu}{}^{\sigma}-\eta_{\nu[\kappa}\eta_{\lambda]}{}^{\rho}\eta^{\mu\sigma}\right)\tilde{B}_{\rho\sigma}\,. \tag{10.3}$$

Proceeding from this formula, the irreducible part of the inner curvature that is simplest to evaluate in terms of the analogously irreducible parts K_ρ and $C_{\lambda\mu}{}^{\nu}$ of the second fundamental tensor $K_{\mu\nu}{}^{\rho}$ is of course the inner Ricci scalar, which [subject to (10.1)] works out as

$$R=\frac{p-1}{n-2}\left(\eta^{\rho\sigma}B_{\rho\sigma}-\frac{p}{n-1}B\right) + \frac{p-1}{p}K^{\sigma}K_{\sigma}-C_{\lambda\mu}{}^{\nu}C^{\lambda\mu}{}_{\nu} \tag{10.4}$$

(this being the quantity whose surface integral in the special case $p=2$ gives the ordinary Gauss–Bonnet type invariant that was mentioned at the end of section 8), while the full inner Ricci tensor is given by a slightly longer chain of assorted terms having the form

$$R_{\mu\nu}=\frac{p-2}{n-2}\eta_\mu{}^\rho\eta_\nu{}^\sigma B_{\rho\sigma}$$
$$+\left[\frac{1}{n-2}\left(\eta^{\rho\sigma}B_{\rho\sigma}-\frac{p-1}{n-1}B\right)+\frac{p-1}{p^2}K^\sigma K_\sigma\right]\eta_{\mu\nu}$$
$$+\frac{p-2}{p}C_{\mu\nu}{}^\sigma K_\sigma-C_\mu{}^{\rho\sigma}C_{\nu\rho\sigma}\,. \tag{10.5}$$

For cases in which the imbedded surface has dimension $p\leqslant 3$, as must always be the case in an ordinary four-dimensional spacetime background, the specification of the Ricci contribution provides all that is needed to specify the complete inner curvature tensor. However, to fully specify $R_{\kappa\lambda}{}^\mu{}_\nu$ in higher-dimensional cases for which the imbedded surface has dimension $p\geqslant 4$ it will also be necessary to account for the generically non-zero conformal curvature term $C_{\kappa\lambda}{}^\mu{}_\nu$ that will contribute to the total as given by (8.10). The rather greater algebraic effort required to work out this inner conformal curvature contribution is rewarded by the qualitatively tidy form of the result, which [in contrast with the miscellaneous form of the terms assembled in (10.4) and (10.5)] is homogeneously quadratic in the conformation tensor alone, the contributions of the trace vector K^μ and of the background Ricci tensor $B_{\mu\nu}$ again [as in (9.7)] being found to miraculously cancel out altogether, leaving

$$C_{\kappa\lambda}{}^{\mu\nu}=2C_{[\kappa}{}^{\mu\sigma}C_{\lambda]}{}^\nu{}_\sigma-\frac{4}{p-2}C^{\rho[\mu}{}_\sigma\eta^{\nu]}{}_{[\kappa}C_{\lambda]\rho}{}^\sigma$$
$$-\frac{2}{(p-2)(p-1)}\eta_{[\kappa}{}^\mu\eta_{\lambda]}{}^\nu C_{\rho\sigma}{}^\tau C^{\rho\sigma}{}_\tau\,. \tag{10.6}$$

In view of the theorem [3] recapitulated in section 7, we can therefore draw the memorable conclusion that vanishing of the conformation tensor $C^{\mu\nu}{}_\rho$ is a sufficient condition not only for (local) outer flatness but also for (local) internal conformal flatness, at least for an imbedded surface with dimension $p>3$.

To see that this result still holds for the lower-dimensional cases that are in practice of greatest interest a little more work is required. In the case of a curve, $p=1$, the internal curvature is of course identically zero, and in the case $p=2$ (that of a string) the internal curvature is fully determined by the Ricci scalar alone, (local) conformal flatness holding automatically, the conformal curvature tensor being undefinable. This leaves as the only non-trivial case still to be dealt with that for which the imbedded surface has dimension $p=3$ (which applies to the world sheet of a membrane), and for which the conformal tensor (10.6) is well defined but identically zero, so that to obtain a sufficient condition for internal conformal flatness we need to evaluate the higher derivative tensor $\tilde{R}_{\lambda\mu\nu}$ as given by (8.12). The algebraic effort required to do this is even greater than in the previous case, but as before it is rewarded by a very neat result. After the rather miraculous cancellation of many contributions involving $B_{\mu\nu}$ and K^ρ one is left just with a homogeneous bilinear combination only of the conformation tensor and of the third fundamental tensor, whose details are expressible as

$$\tilde{R}_{\lambda\mu\nu} = C_\nu{}^{\rho\sigma}\Xi_{[\mu\lambda]\rho\sigma} + C_{\nu[\mu}{}^\sigma\Xi_{\lambda]\rho}{}^\rho{}_\sigma - C_{\rho[\mu}{}^\sigma\Xi_{\lambda]\nu}{}^\rho{}_\sigma$$

$$+ \frac{1}{p-1}\eta_{\nu[\mu}C^{\sigma\rho\tau}\Xi_{\lambda]\sigma\rho\tau} \tag{10.7}$$

(in which the first term on the right will vanish by the strong symmetry condition (6.7) whenever the background is not just conformally flat but strictly flat or of constant curvature). Since it evidently follows that if $C_{\mu\nu}{}^\rho$ is zero then so is $\tilde{R}_{\lambda\mu\nu}$, we thus complete the demonstration that the vanishing of the trace free conformation tensor of the imbedding is always, without exception, a sufficient condition for conformal flatness in a conformally flat background.

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Appendix A. Background tensor surface forms and Stokes theorem

It is evident that there is a natural bijective correspondence between the intrinsically defined tensors *within* a given spacelike or timelike p-dimensional submanifold $\mathscr{S}_p$ with first fundamental tensor $\eta^{\mu\nu}$, say, and the set of *projection invariant* background tensors, i.e., those that are invariant under the projection operation – whose effect we shall denote by an *overhead bar* – consisting of contraction of all indices with the corresponding projection tensor, which, for action on a tensor with q indices, will be given by

$$\eta^{\nu_1\cdots\nu_q}_{\mu_1\cdots\mu_q} = \eta_{\mu_1}{}^{\nu_1}\cdots\eta_{\mu_q}{}^{\nu_q}\,, \tag{A.1}$$

which is the analogue, with respect to projection invariant q-index tensors on the p-surface, of the ordinary Kronecker operator

$$\delta^{\nu_1\cdots\nu_q}_{\mu_1\cdots\mu_q} = \delta_{\mu_1}{}^{\nu_1}\cdots\delta_{\mu_q}{}^{\nu_q}\,, \tag{A.2}$$

that acts as an identity operator on a general q-index background tensor.

Let us focus attention on the case of an arbitrary smooth q-dimensional subsurface $\mathcal{S}_q$, say, that we suppose to be confined within the non-null p-surface $\mathcal{S}_p$ with respect to which the projection operator (A.1) has been defined:

$$\mathcal{S}_q \subset \mathcal{S}_p . \tag{A.3}$$

The only kind of tensor $\boldsymbol{F}$ whose integral over an imbedded q-surface is naturally well defined independently of any auxiliary (e.g. metric or linear) structure is (as has been well known since the time of Cartan) that of a q-form, meaning that it must be fully antisymmetric and covariant with components

$$F_{\mu_1 \cdots \mu_q} = F_{[\mu_1 \cdots \mu_q]} . \tag{A.4}$$

Its projection $\bar{\boldsymbol{F}}$ with components

$$\bar{F}_{\mu_1 \cdots \mu_q} = \eta^{\nu_1 \cdots \nu_q}_{\mu_1 \cdots \mu_q} F_{\nu_1 \cdots \nu_q} \tag{A.5}$$

will therefore also be a q-form on $\mathcal{S}_p$, even though it is undefined elsewhere on the background. What is important to notice is that subject to (A.3) the natural (Cartan) integral of $\boldsymbol{F}$ is necessarily the *same* as that of its projection $\bar{\boldsymbol{F}}$ as given by (A.5), i.e., in standard shorthand notation

$$\int_{\mathcal{S}_q} \boldsymbol{F} = \int_{\mathcal{S}_q} \bar{\boldsymbol{F}} . \tag{A.6}$$

The most important basic result in the Cartan integral calculus is the generalised Stokes theorem:

$$\boldsymbol{F} = \partial \boldsymbol{A} \quad \Rightarrow \quad \int_{\mathcal{S}_q} \boldsymbol{F} = \oint_{\mathcal{S}_{q-1}} \boldsymbol{A} , \tag{A.7}$$

whenever $\mathcal{S}_q$ has compact closure with a smooth compact $(q-1)$-dimensional boundary manifold $\mathcal{S}_{q-1}$, where $\partial \boldsymbol{A}$ denotes the *exterior derivative* of a $(q-1)$-form $\boldsymbol{A}$, which is definable by an expression of the form

$$(\partial \boldsymbol{A})_{\lambda \mu_2 \cdots \mu_q} = q \nabla^{\nu_1 \nu_{q-1}}_{\mu_2 \cdots \mu_q \lambda} A_{\nu_1 \cdots \nu_{q-1}} = q \nabla_{[\lambda} A_{\mu_2 \cdots \mu_q]} , \tag{A.8}$$

where the components of the *antisymmetrised derivation* operator are specified by

$$\nabla^{\nu_q \cdots \nu_{q-1}}_{\mu_1 \cdots \mu_{q-1} \mu_q} = \delta^{\nu_1 \cdots \nu_{q-1}}_{[\mu_1 \cdots \mu_{q-1}} \nabla_{\mu_q]} = \delta^{\nu_1 \cdots \nu_{q-1} \lambda}_{[\mu_1 \cdots \mu_{q-1} \mu_q]} \nabla_\lambda . \tag{A.9}$$

The exterior derivation operator (A.8) is well known to have the Poincaré exactness property (on which cohomology theory is based), meaning that the "closure" of $\boldsymbol{F}$, i.e., the vanishing of its own exterior derivative, is both necessary and locally sufficient for it to be the exterior derivative of a $(q-1)$-form $\boldsymbol{A}$:

$$\partial \partial \boldsymbol{F} = 0 \quad \Leftrightarrow \quad \boldsymbol{F} = \partial \boldsymbol{A} . \tag{A.10}$$

It is evident that the general exterior derivation operation ∂ defined by (A.8) is meaningful only for a form $\boldsymbol{A}$ that is defined on an open neighbourhood on the background space, but not for a form whose support is confined to a lower-dimensional submanifold. In order to be able to deal in an analogous way with forms whose support is confined to the p-dimensional non-null submanifold $\mathscr{S}_p$ with fundamental tensor $\eta_\mu{}^\nu$ that was introduced above, it will, however, suffice to work with the corresponding *projected exterior derivation* operation $\bar{\partial}$ that is naturally defined by

$$(\bar{\partial}\boldsymbol{A})_{\lambda\mu_2\cdots\mu_q}=q\bar{\nabla}^{\nu_1\cdots\nu_{q-1}}_{\mu_2\cdots\mu_q\lambda}A_{\nu_1\cdots\nu_{q-1}}\,, \tag{A.11}$$

where the components of the *projected antisymmetrised derivation* operator are specified by

$$\bar{\nabla}^{\nu_1\cdots\nu_{q-1}}_{\mu_1\cdots\mu_{q-1}\mu_q}=\eta^{\nu_1\cdots\nu_{q-1}}_{[\mu_1\cdots\mu_{q-1}}\bar{\nabla}_{\mu_q]}=\eta^{\nu_1\cdots\nu_{q-1}\lambda}_{[\mu_1\cdots\mu_{q-1}\mu_q]}\nabla_\lambda\,. \tag{A.12}$$

As thus defined, the operations of projection and exterior derivation *commute* in the sense that [as can easily be checked explicitly using the Weingarten symmetry property (5.4)] the projection of an ordinary exterior derivative gives the same result as the action of the projected exterior derivative on the direct projection, i.e., for any background q-form $\boldsymbol{F}$ defined in an open neighbourhood of the imbedded p-surface $\mathscr{S}_p$ we have the identity

$$\overline{(\partial\boldsymbol{F})}=\bar{\partial}\bar{\boldsymbol{F}}\,. \tag{A.13}$$

Although it is well defined only on $\mathscr{S}_p$, the projected exterior derivation operation has the advantage of still being well defined there even when acting on a form $\boldsymbol{F}$ whose support is already confined to $\mathscr{S}_p$, in which case the left hand side of (A.13) would be undefined whereas the right hand side of (A.13) would remain unambiguously meaningful: under the bijective relation mentioned at the beginning of this section, the form $\bar{\partial}\boldsymbol{F}$ is the projection invariant background tensor corresponding to the intrinsically defined exterior derivative within (not just on) $\mathscr{S}_p$ (considered as a p-dimensional manifold in its own right) of the intrinsic form corresponding of the projection invariant background tensor $\bar{\boldsymbol{F}}$. It follows, as a consequence of the intrinsic Poincaré property within $\mathscr{S}_p$ [or equivalently by (A.13) as a consequence of the background Poincaré property (A.10) for a form whose domain of definition extends out into an open neighbourhood on the background space], that we shall get a projected Poincaré property to the effect that the vanishing of the projected exterior derivative of a projection invariant form $\bar{\boldsymbol{F}}$ is both necessary and locally sufficient for it to be the projected exterior derivative of a projection invariant $(q-1)$-form $\bar{\boldsymbol{A}}$, say:

$$\bar{\partial}\bar{\partial}\bar{\boldsymbol{F}}=0 \quad\Leftrightarrow\quad \bar{\boldsymbol{F}}=\bar{\partial}\bar{\boldsymbol{A}}\,. \tag{A.14}$$

Similarly, as a consequence of the intrinsic Stokes theorem within $\mathscr{S}_p$ [or equivalently by (A.13) as a consequence of the background Stokes theorem (A.7) for a form whose domain of definition extends out into an open neighbourhood on the background space], it can be seen [bearing in mind (A.6)] that we shall get a projected Stokes theorem to the effect that, whenever $\mathscr{S}_q$ has compact closure with a smooth compact $(q-1)$-dimensional boundary manifold $\mathscr{S}_{q-1}$ within $\mathscr{S}_p$, we shall have

$$\oint_{\mathscr{S}_{q-1}} \boldsymbol{A} = \int_{\mathscr{S}_q} \bar{\partial}\bar{\boldsymbol{A}} \tag{A.15}$$

for any $(q-1)$-form $\boldsymbol{A}$ with support on $\mathscr{S}_p$. This is interpretable as a strengthening of the original generalised Stokes theorem (A.7), because it applies even if $\boldsymbol{A}$ is not specified outside $\mathscr{S}_p$ and so has no well-defined exterior derivative $\partial\boldsymbol{A}$ of the ordinary kind that appears in the usual formulation (A.7).

Appendix B. Background tensor surface divergence and Green theorem

Rather than working with the covariant forms that are most fundamental from a mathematical point of view, and at the expense of having to introduce an appropriate measure, physicists tend to prefer to use a *dual formulation* whereby the fluxes of interest are treated in terms of *contravariant* current multivectors: clearly any q-form $\boldsymbol{F}$ has a corresponding dual $(n-q)$-vector (i.e. a fully antisymmetric contravariant tensor), $\boldsymbol{\beta}$ say, in terms of which it can be expressed in the form

$$\boldsymbol{F} = \boldsymbol{\beta}_{\bullet}\,, \tag{B.1}$$

where the duality relation is defined by

$$(n-q)!\,\beta_{\bullet\,\mu_1\cdots\mu_q} = \beta^{\nu_1\cdots\nu_{n-q}}\epsilon_{\nu_1\cdots\nu_{n-q}\mu_1\cdots\mu_q}\,, \tag{B.2}$$

in terms of the standard measure form (the ordinary alternating tensor) $\boldsymbol{\epsilon}$ of the n-dimensional background space. (The dual formulation has the technical advantage of reducing the number of indices involved when $2q > n$ so that $n-q$ is smaller than q.)

The dual analogue of the exterior derivation operation indicated here by ∂ is the interior derivative or *divergence* operation that we shall indicate here by the abbreviation div. (In pure mathematical texts it is common practice, in place of ∂ and div, to simply use d and δ, respectively, but we deliberately refrain from doing this here in order to avoid confusion, in view of the many other uses of those much overworked symbols.) This operation is constructed in such a way that for any multivector $\boldsymbol{\beta}$ the identity

$$(\operatorname{div}\boldsymbol{\beta})_{\bullet} = \partial(\boldsymbol{\beta}_{\bullet}) \tag{B.3}$$

is always satisfied, which is done by defining the components of the divergence to be given for an $(r+1)$-vector simply as

$$(\operatorname{div}\beta)^{\nu_1\cdots\nu_r}=\nabla^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_{r+1}}\beta^{\mu_1\cdots\mu_{r+1}}=\nabla_\lambda\beta^{\nu_1\cdots\nu_r\lambda}\,. \tag{B.4}$$

With this convention, the Stokes theorem (A.7) can be rewritten as

$$\boldsymbol{\beta}=\operatorname{div}\boldsymbol{J}\quad\Rightarrow\quad\int_{\mathscr{S}_q}\boldsymbol{\beta}_{\star}=\oint_{\mathscr{S}_{q-1}}\boldsymbol{J}_{\star}\,, \tag{B.5}$$

this dual reformulation being what is commonly known as Green's theorem.

Although it commutes with exterior derivation in the sense expressed by (A.13), the p-surface projection operation does *not* have an analogous commutation relation with the duality operation (B.2). Thus there is no simple analogue of (A.12) for divergences, while similarly there is no simple dual analogue of (A.6), i.e., it is not generally possible to replace an r-vector $\boldsymbol{J}$ by its projection $\bar{\boldsymbol{J}}$ as given by

$$\bar{J}^{\nu_1\cdots\nu_r}=\bar{J}^{[\nu_1\cdots\nu_r]}=\eta^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_r}J^{\mu_1\cdots\mu_r} \tag{B.6}$$

in a dual integral expression such as that on the right of (B.5).

The foregoing caveat means that more care is needed to deal with divergences in an imbedding than with the more fundamental exterior type of derivation operation, but it does not mean that there is any obstacle to performing the intrinsic analogue for an imbedded p-surface of the usual trick whereby the Stokes theorem as expressed in terms of exterior derivation is transformed into the Green theorem involving interior derivation. To perform such a trick, so as to obtain a dual analogue of (A.15) in terms of a projection invariant multivector field $\boldsymbol{J}$ say (i.e., one for which $\bar{\boldsymbol{J}}=\boldsymbol{J}$), it will, however, be necessary to go back and start again with a modified kind of duality, which we shall indicate by the use of a five instead of six pointed star, that is defined in terms of the restricted p-surface analogue of the background measure tensor, namely the fully antisymmetric projection invariant p-form that is fully characterised modulo a choice of orientation by

$$\mathscr{E}_{\mu_1\cdots\mu_p}=\bar{\mathscr{E}}_{[\mu_1\cdots\mu_p]}\,, \tag{B.7}$$

together with the standard normalisation condition that is fixed by any single one of the set of mutually consistent self-contraction conditions

$$\mathscr{E}^{\kappa_1\cdots\kappa_q\mu_{q+1}\cdots\mu_p}\mathscr{E}_{\lambda_1\cdots\lambda_q\mu_{q+1}\cdots\mu_p}=\pm q!(p-q)!\eta^{\kappa_1\cdots\kappa_q}_{[\lambda_1\cdots\lambda_q]}\,, \tag{B.8}$$

giving (on the right) the antisymmetrisation of the projection operator [as defined by (A.1)] for any value of q in the range from 0 to p, the sign $\pm$ being the signature of the induced metric on the p-surface. It can be seen that the covariant surface gradient of this tensor will be expressible in terms of the second fundamental tensor of the p-surface in the form

$$\bar{\nabla}_\lambda\mathscr{E}^{\mu_1\cdots\mu_p}=p\mathscr{E}^{\nu[\mu_1\cdots}K_{\lambda\nu}{}^{\mu_p]}\,. \tag{B.9}$$

In terms of the standard surface measure p-form (B.7), the projected surface analogue $\bar{\boldsymbol{\beta}}_{\star}$ of the ordinary dual $\boldsymbol{\beta}_{*}$ for an r-vector $\boldsymbol{\beta}$ is naturally definable by the obvious analogue of (B.2) as

$$\beta_{\star\mu_{r+1}\cdots\mu_p}=\frac{1}{(p-q)!}\beta^{\nu_1\cdots\nu_r}\mathscr{E}_{\nu_1\cdots\nu_r\mu_{r+1}\cdots\mu_p}\,. \tag{B.10}$$

This definition evidently satisfies identities of the form

$$\boldsymbol{\beta}_{\star}=\overline{(\boldsymbol{\beta}_{\star})}=(\bar{\boldsymbol{\beta}})_{\star} \tag{B.11}$$

(which would fail to hold if $*$ were substituted in place of $\star$). It can be seen [with the aid of (B.9)] that this definition is such that we recover the natural surface projected analogue of the identity (B.3) in the form

$$(\overline{\mathrm{div}}\,\boldsymbol{\beta})_{\star}=\bar{\partial}(\bar{\boldsymbol{\beta}}_{\star})\ , \tag{B.12}$$

where the projected derivation operator $\overline{\mathrm{div}}$ is defined by the obvious analogue of (B.4) in terms of the projected antisymmetrised derivation operator (A.12) according to the specification

$$(\overline{\mathrm{div}}\ \beta)^{\nu_1\cdots\nu_r}=\bar{\nabla}^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_{r+1}}\beta^{\mu_1\cdots\mu_{r+1}}\,. \tag{B.13}$$

It follows from (B.12) that we get a dual reformulation of the projected Poincaré property (A.10), to the effect that the vanishing of the projected divergence of a projection invariant r-vector $\bar{\boldsymbol{\beta}}$ is both necessary and locally sufficient for it to be the projected divergence of a projection invariant $(r+1)$-vector $\bar{\boldsymbol{J}}$, say:

$$\overline{\mathrm{div}}\ \overline{\mathrm{div}}\,\bar{\boldsymbol{\beta}}=0\quad\Leftrightarrow\quad\bar{\boldsymbol{\beta}}=\overline{\mathrm{div}}\,\bar{\boldsymbol{J}}\,. \tag{B.14}$$

Finally, with the definitions (B.11) and (B.13) we also get the natural dual reformulation of the projected Stokes theorem (A.15) as a *surface projected Green theorem* stating that, whenever $\mathscr{S}_q$ has compact closure with a smooth compact $(q-1)$-dimensional boundary manifold $\mathscr{S}_{q-1}$ within $\mathscr{S}_p$, we shall have

$$\oint_{\mathscr{S}_{q-1}}\boldsymbol{J}_{\star}=\int_{\mathscr{S}_q}(\overline{\mathrm{div}}\,\bar{\boldsymbol{J}})_{\star} \tag{B.15}$$

for any $(p-q+1)$-vector $\boldsymbol{J}$ with support on $\mathscr{S}_p$.

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